Homework 2

Math 117 - Summer 2022

1) Let $V = \mathbb{Q}_3[x]$ and let $W = \mathbb{Q}_2[x]$. Consider the transformation $T : V \to W$ defined by

$$T(f) = 3\frac{d}{dx}f - 2(x+1)\frac{d^2}{dx^2}f$$

(a) (1 point) Compute the matrix $[T]_{\mathcal{B}}^{\mathcal{C}}$ with respect to the bases $\mathcal{B} = (1, x, x^2, x^3)$ and $\mathcal{C} = (1, x, x^2)$

(b) (1 point) Compute the matrix $[T]_{\mathcal{B}'}^{\mathcal{C}'}$ with respect to the basis $\mathcal{B}' = (1, x+1, x^2+1, x^3+1)$ and $\mathcal{C} = (1, x+1, x^2+1)$

(c) (1 point) Compute the change of basis matrix P from $\mathcal{B}$ to $\mathcal{B}'$

(d) (1 point) Compute the change of basis matrix Q from $\mathcal{C}$ to $\mathcal{C}'$

(e) (2 points) Compute P^{-1} and verify the formula $[T]_{\mathcal{B}'}^{\mathcal{C}'} = Q[T]_{\mathcal{B}}^{\mathcal{C}}P^{-1}$

Solution:

2) (4 points) Let V be an $\mathbb{F}$ vector space and let $V_1, \ldots, V_n$ be a subspaces of V. The point of this problem is to show the "internal" and "external" definitions of direct sum are equivalent. That is: prove the following are equivalent

(a) $V = V_1 + \cdots + V_n$ and $V_i \cap \sum_{j \neq i} V_j = \{0_V\}$

(b) Every vector $v \in V$ has a unique expression $v = v_1 + \cdots + v_n$ with $v_i \in V_i$

(c) $V = V_1 + \cdots + V_n$ and the map

$$\pi : V_1 \oplus \cdots \oplus V_n \to V_1 + \cdots + V_n$$
$$\pi(v_1, \ldots, v_n) = v_1 + \cdots + v_n$$

is an isomorphism

Can we replace the sum condition in (a) by just asking that $V_i \cap V_i = \{0_V\}$ for all $i \neq j$?

Solution:

3) The point of this problem is to show that linear maps out of/into direct sums "behave" like matrices. Let $V_1, \ldots, V_n, W_1, \ldots, W_m$ be vector spaces over $\mathbb{F}$.

(a) (2 points) Suppose we are given linear maps $T_{ij} : V_j \to W_i$ for each i,j. Construct a linear map

$$T : V_1 \oplus \cdots \oplus V_n \to W_1 \oplus \cdots \oplus W_m$$

that "behaves" like the $m \times n$ matrix of linear maps

$$\begin{pmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & T_{2n} \\ \dots & \dots & \dots & \dots \\ T_{m1} & T_{m2} & \dots & T_{mn} \end{pmatrix}$$

(b) (2 points) Show that every linear map

$$T : V_1 \oplus \cdots \oplus V_n \to W_1 \oplus \cdots \oplus W_m$$

is of this form for linear maps $T_{ij} : V_j \to W_i$ (hint: inclusion into direct sum/projection onto subspace)

(c) (1 point) Conclude that the set of linear maps

$$T : V_1 \oplus \cdots \oplus V_n \to W_1 \oplus \cdots \oplus W_m$$

can be identified with the set of matrices

$$\begin{pmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & T_{2n} \\ \dots & \dots & \dots & \dots \\ T_{m1} & T_{m2} & \dots & T_{mn} \end{pmatrix}$$

with coefficients $T_{ij} \in \mathcal{L}(V_j, W_i)$

(d) (1 point) Does anything familiar happen when $V_1 = \cdots = V_n = W_1 = \cdots = W_m = \mathbb{F}$? (Hint: when $V_j = W_i = \mathbb{F}$ what is $\mathcal{L}(V_j, W_i)$ isomorphic to? Think dimensions....)

Solution:

Unimportant but cool remark: The above discussion is actually saying something quiet deep and general. In Category theory there is a notion of what is called a "categorical product" which can be thought of as "the best approximation of a collection of objects above the objects." More concretely, the product (if it exists) of a collection of objects $(X_i)_{i \in I}$ is an object $\prod X_i$ equipped with maps, called projections, $\pi_i : \prod X_i \to X_i$ for each i that are somehow "universal" (or the "best" possible maps). In the Category of finite dimensional vector spaces, the categorical product of vector spaces $V_1, \dots, V_n$ is just the Cartesian product of them $V_1 \times \cdots \times V_n$!!!

Now, there is a slightly less intuitive notion of what is called a "categorical coproduct" which is basically "the best approximation of a collection of objects below the objects." Given a collection of objects $(X_i)_{i \in I}$ the coproduct is denoted (if it exists) by $\coprod X_i$. It again comes equipped with a collection of maps, this time going into it $\iota_i : X_i \to \coprod X_i$ that is again somehow "universal"

Sometimes, something very special happens, and we have an isomorphism $\prod X_i \simeq \coprod X_i$. In this case, we call this object the "biproduct" and denote it by $\bigoplus X_i$. The notation probably gives it away but yes! It turns out in the category of vector spaces $\prod_{i=1}^n V_i \simeq \coprod_{i=1}^n V_i$ which we call the direct sum $\bigoplus_{i=1}^n V_i$

What we saw above in this previous problem can thus be phrased as: "maps between biproducts can be expressed as "matrices" of maps." This is true in general!!! In any category where the biproduct of objects exists, maps between them behave like matrices of maps- this turns out to be quiet useful

4) (4 points) Let V be finite dimensional vector space and let $W \subset V$ be subspace with complimentary subspace W'. Prove that the composite

$$W' \hookrightarrow V \to V/W$$

is an isomorphism

Solution: